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Subdifferentials of a minimal time function in normed spaces ☆

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ABSTRACT

In a general normed vector space, we study the minimal time function determined by a differential inclusion where the set-valued mapping involved has constant values of a bounded closed convex set U and by a closed target set S. We show that proximal and Fréchet subdifferentials of a minimal time function are representable by virtue of corresponding normal cones of sublevel sets of the function and level or suplevel sets of the support function of U. The known results in the literature require the set U to have the origin as an interior point or U be compact. (In particular, if the set U is the unit closed ball, the results obtained reduce to the subdifferential of the distance function defined by S.)

1. Introduction

Let X be a normed vector space and U be a bounded closed convex subset of X. Let S be a closed subset of X. The function T_S is defined by

$$T_S(x) := \inf\{t \geqslant 0\colon\ S \cap (x + tU) \neq \emptyset\}, \quad \text{for all } x \in X. \tag{1.1}$$

It can be seen that $T_S(x)$ is the minimal time function defined by the following differential inclusion

$$\dot{x}(t) \in U, \qquad x(0) = x. \tag{1.2}$$

In other words,

$$T_S(x) \equiv \begin{cases} \inf\{T > 0\colon \text{there exists a trajectory } x(\cdot) \text{ satisfying (1.2) with } x(0) = x \text{ and } x(T) \in S\}, & x \notin S; \\ 0, & x \in S. \end{cases}$$

Various properties of the minimal time function have been studied in the literature; see [1–6]. Since this function is not necessarily convex, some researchers discussed its subdifferential in the sense of nonsmooth analysis.

It can be seen that the minimal time function T_S is a generalization of the usual distance function, and so the study of the subdifferential of T_S extends the subdifferential calculus of the usual distance. On the other hand, consider the following

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problem

$$\begin{aligned} \min \quad & \varphi(t,x) \\ \text{s.t.} \quad & t \in A \\ & x \in F(t), \end{aligned} \tag{1.3}$$

where φ is a function, A is a set, and F is a set-valued mapping. The Fréchet subdifferential of the optimal value function of (1.3) was studied in [7, Section 3]: An upper estimate of the Fréchet subdifferential of the value function of (1.3) is given in terms of the normal cone of A and the coderivative of the set-valued mapping F. Obviously, the minimal time function $x \mapsto T_S(x)$ is the optimal value function of (1.3) where $\varphi(t,x) = t$, $A = \mathbb{R}_+$, and $F(t) = S - tU$. In fact, the study of subdifferential of optimal value function has been very important in the area of optimization, especially in the theory of Lagrange multipliers and in stability and sensitivity analysis; see [8–10] and the references therein. As a byproduct, our results show that for T_S, an exact equality of the subdifferential holds. Thus the main results refine the known results on the subdifferential of value function by replacing the upper estimate by an equality.

Let us recall more backgrounds. Assuming that the origin is an interior point of U, Colombo and Wolenski [2,3] studied the proximal and Fréchet subdifferentials of the function $T_S(x)$ in a Hilbert space. [11] studied the Fréchet and proximal subdifferentials of T_S in a Banach space. When the origin is an interior point of U, the function T_S is globally Lipschitz, so the Clarke subdifferential of T_S is also discussed in [11]. In particular, if U is the (closed) unit ball in X, then $T_S(x)$ reduces to the usual distance $d_S(x)$, which is defined by

$$d_S(x) := \inf_{s \in S} \|s - x\|, \quad \text{for all } x \in X.$$

The subdifferentials of d_S were studied in [12–14].

In this paper, we do not require the origin be an interior point of U and show that the Frechét and proximal subdifferentials of the minimal time function T_S can be described by virtue of the corresponding notions of normal cones of sublevel sets of T_S and the support function of U. The space is assumed to be a normed vector space of possibly infinite dimension. The main results unify and generalize the corresponding results in [12–14,2,3]: the set U does not necessarily contain the origin as an interior point, actually U can have empty interior; the set U is not necessarily compact; the space X is a general normed vector space. A simple example shows that the boundedness assumption of U is indispensable.

2. Preliminaries

Let X be a normed vector space with norm denoted by $\|\cdot\|$. Let X^* denote the topological dual of X. We use $B(x;r)$ to denote the open ball centered at x with radius $r > 0$ and $\langle\cdot,\cdot\rangle$ to denote the pairing between X^* and X. Let $g : X \to \mathbb{R}$ be a lower semicontinuous function and $x \in X$. g is said to be *calm* at x [15] if there exist $k > 0$ and a neighborhood V of x such that

$$\bigl|g(y) - g(x)\bigr| \leqslant k\|y - x\|, \quad \forall y \in V. \tag{2.1}$$

In particular, if g is locally Lipschitz at x, then it is calm at x.

Let us recall the following well-known classes of subdifferentials for g at x.

- The *proximal subdifferential* of g at x is the set

$$\partial^P g(x) := \left\{\xi \in X^*:\ \liminf_{\|v\|\to 0} \frac{g(x+v) - g(x) - \langle \xi, v\rangle}{\|v\|^2} > -\infty\right\}.$$

In other words, $\xi \in \partial^P g(x)$ if and only if there exist $\sigma > 0$ and $\delta > 0$ such that

$$g(x+v) - g(x) \geqslant \langle \xi, v\rangle - \sigma\|v\|^2, \quad \text{for all } v \in B(0,\delta).$$

- The *Frechét subdifferential* of g at x is the set

$$\partial^F g(x) := \left\{\xi \in X^*:\ \liminf_{\|v\|\to 0} \frac{g(x+v) - g(x) - \langle \xi, v\rangle}{\|v\|} \geqslant 0\right\}.$$

Let $S \subset X$ be a closed set and let $x \in S$. The *proximal normal cone* and *Frechét normal cone* of S at x are defined as the corresponding subdifferential of the indicator function of S at x and are denoted respectively as $N_S^P(x)$ and $N_S^F(x)$. That is, $\xi \in N_S^P(x)$ if and only if there exist $\sigma > 0$ and $\delta > 0$ such that $\langle \xi, y - x\rangle \leqslant \sigma\|y - x\|^2$ for all $y \in S \cap B(x;\delta)$, and $\xi \in N_S^F(x)$ if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\langle \xi, y - x\rangle \leqslant \varepsilon\|y - x\|$ for all $y \in S \cap B(x;\delta)$.

The support function of a set $K \subset X$ is defined by

$$\Im_K(\xi) := \sup_{x \in K} \langle \xi, x\rangle.$$

Proposition 2.1. *If $x \notin S$,*

$$T_S(x) = \inf\{t > 0\colon\ S \cap (x + tU) \neq \emptyset\}; \tag{2.2}$$

if $x \in S$ and if U contains the origin, then (2.2) *still holds.*

Proposition 2.2. *Let U be a nonempty bounded set in X and $S \subset X$ be a nonempty closed set. Then*

(i) *$T_S(x) = 0$ if and only if $x \in S$.*
(ii) *If U is compact and if $T_S(x) < \infty$, then the infimum defining $T_S(x)$ is attained.*

Proof. It is obvious that $T_S(x) = 0$ for $x \in S$. Conversely, if $T_S(x) = 0$, by the definition of T_S, there is a nonnegative scalar sequence $\{t_n\}$ such that $t_n \to 0+$ and $S \cap (x + t_n U) \neq \emptyset$. This implies the existence of $\{u_n\} \subset U$ satisfying that $\{x + t_n u_n\} \subset S$. Since $t_n \to 0+$ and U is bounded, it follows that $x + t_n u_n \to x$, and hence $x \in S$ as S is closed.

To verify (ii), let $\{t_n\}$ be a minimizing sequence: $t_n \geqslant 0$, $t_n \to T_S(x)$, and $S \cap (x + t_n U) \neq \emptyset$. Then there exists $u_n \in U$ such that $x + t_n u_n \in S$. Since U is compact, we may assume that $u_n \to u$ for some $u \in U$. Thus $x + t_n u_n \to x + T_S(x)u$. The closedness of S shows that $x + T_S(x)u \in S$. $\square$

Remark 2.1. It is possible that $T_S(x) = \infty$ for some x. For example, when $X = \mathbb{R}^2$, $S = \{0\}$, $x = (0, 1)$, and $U = [-1, 1] \times \{0\}$.

3. Proximal subdifferential of a minimal time function

Theorem 3.1. *Let $x \in S$. Then*

$$\partial^P T_S(x) = N_S^P(x) \cap \{\xi \in X^*\colon\ \Im_U(-\xi) \leqslant 1\}.$$

Proof. Let $x \in S$ and $\xi \in \partial^P T_S(x)$. Then there exist $\sigma, \delta > 0$ such that

$$T_S(y) - T_S(x) - \langle \xi, y - x\rangle \geqslant -\sigma \|y - x\|^2, \tag{3.1}$$

for all $y \in B(x; \delta)$. It follows from Proposition 2.2 that for all $y \in S \cap B(x; \delta)$,

$$\langle \xi, y - x\rangle \leqslant \sigma \|y - x\|^2. \tag{3.2}$$

Therefore $\xi \in N_S^P(x)$.

Let $v \in U$ and $t_\lambda := T_S(x - \lambda v)$, where $\lambda > 0$. Then

$$x \in S \cap (x - \lambda v + \lambda U) \neq \emptyset,$$

and hence $t_\lambda \leqslant \lambda < \infty$. It follows from (3.1) that for sufficiently small $\lambda > 0$,

$$\lambda \geqslant t_\lambda \geqslant \lambda\langle -\xi, v\rangle - \lambda^2 \sigma \|v\|^2,$$

which implies that $\langle -\xi, v\rangle \leqslant 1$. Therefore, $\Im_U(-\xi) \leqslant 1$.

Conversely, let $\xi \in N_S^P(x)$ be such that $\Im_U(-\xi) \leqslant 1$. Then there exist $\sigma_1, \delta > 0$ such that

$$\langle \xi, y - x\rangle \leqslant \sigma_1 \|y - x\|^2, \quad \forall y \in S \cap B(x; \delta). \tag{3.3}$$

Put $\sigma := 2(M^2\|\xi\|^2 + 1)\sigma_1$, where $M := \sup_{u \in U} \|u\|$. In view of Proposition 2.2, $T_S(y) = 0$ for $y \in S$. Thus (3.3) implies that

$$T_S(y) - T_S(x) - \langle \xi, y - x\rangle \geqslant -\sigma \|y - x\|^2,$$

for all $y \in S \cap B(x; \delta)$. Let $\delta_0 := \frac{\delta}{2(1 + M\|\xi\|)}$. Now we prove that the above inequality holds for all $y \in B(x; \delta_0) \setminus S$. Therefore, $\xi \in \partial^P T_S(x)$. If not, then there is $y_0 \notin S$ such that

$$\|y_0 - x\| < \delta_0 \quad \text{and} \quad T_S(y_0) < \langle \xi, y_0 - x\rangle - \sigma \|y_0 - x\|^2. \tag{3.4}$$

The latter implies that

$$T_S(y_0) \leqslant \|\xi\|\|y_0 - x\|. \tag{3.5}$$

Let $\theta_0 := T_S(y_0)$. Then (3.5) implies that $\theta_0 < \infty$. By the definition of T_S, for any $\varepsilon \in (0, \frac{\delta}{2M+1})$, there are $t_1 \in (\theta_0, \theta_0 + \varepsilon)$, $s \in S$, and $u \in U$ such that $s = y_0 + t_1 u$. Thus

$$\|s-x\| \leqslant \|y_0-x\|+t_1\|u\| \leqslant \|y_0-x\|+(\theta_0+\varepsilon)M \tag{3.6}$$
$$\leqslant \big(1+M\|\xi\|\big)\|y_0-x\|+\varepsilon M$$
$$< \big(1+M\|\xi\|\big)\delta_0+\varepsilon M < \delta, \tag{3.7}$$

where the third inequality follows from (3.5). This verifies that $s \in S \cap B(x;\delta)$. It follows from (3.3) and (3.6) that

$$T_S(y_0)-\langle\xi, y_0-x\rangle = \theta_0-\langle\xi, y_0-s\rangle-\langle\xi, s-x\rangle \geqslant \theta_0-\langle\xi, y_0-s\rangle-\sigma_1\|s-x\|^2 = \theta_0+t_1\langle\xi,u\rangle-\sigma_1\|s-x\|^2$$
$$\geqslant \theta_0-t_1-\sigma_1\|s-x\|^2 \geqslant -\varepsilon-\sigma_1\|s-x\|^2 \geqslant -\varepsilon-2\sigma_1M^2(\theta_0+\varepsilon)^2-2\sigma_1\|y_0-x\|^2.$$

Letting $\varepsilon \to 0+$, we obtain

$$T_S(y_0)-\langle\xi, y_0-x\rangle \geqslant -2\sigma_1M^2\theta_0^2-2\sigma_1\|y_0-x\|^2 \geqslant -2\sigma_1\big(M^2\|\xi\|^2+1\big)\|y_0-x\|^2 = -\sigma\|y_0-x\|^2,$$

where the second inequality follows from (3.5). This contradicts to the second expression in (3.4). □

Example 3.1. Let $U=\{0\}$. Then it can be seen that $T_S(x)=0$ for $x \in S$ and $T_S(x)=\infty$ for $x \notin S$. That is, $T_S(x)$ is the indicator function of the set S. By Theorem 3.1, for $x \in S$,

$$\partial^P T_S(x) = N_S^P(x),$$

which reduces to an equivalent definition of proximal normal cone.

Example 3.2. Let $X := \mathbb{R}^2$, $U := [-1,1]\times\{0\}$, and S be the unit ball in $\mathbb{R}^2$. Then

$$T_S(x) = \begin{cases} 0, & x \in S, \\ |x_1|-\sqrt{1-x_2^2}, & x \notin S \text{ and } |x_2| \leqslant 1, \\ \infty, & \text{otherwise}. \end{cases}$$

At $x=(1,0)$, $\partial^P T_S(x) = [0,1]\times\{0\}$.

Theorem 3.2. *Let $x \notin S$ and $r := T_S(x) < \infty$. We have*

(a) $\partial^P T_S(x) \subset N_{S(r)}^P(x) \cap \{\xi \in X^*\colon\ \Im_U(-\xi)=1\}$;
(b) *If T_S is calm at x, then*

$$N_{S(r)}^P(x) \cap \big\{\xi \in X^*\colon\ \Im_U(-\xi)=1\big\} \subset \partial^P T_S(x).$$

Proof. (a) Let $\xi \in \partial^P T_S(x)$. Then there exist $\sigma, \delta > 0$ such that

$$T_S(y)-T_S(x)-\langle\xi, y-x\rangle \geqslant -\sigma\|y-x\|^2, \tag{3.8}$$

for all $y \in B(x;\delta)$. Since $T_S(\cdot) \leqslant r$ on $S(r)$ and $r=T_S(x)$, it follows that

$$\langle\xi, y-x\rangle \leqslant \sigma\|y-x\|^2, \tag{3.9}$$

for all $y \in S(r)\cap B(x;\delta)$, that is, $\xi \in N_{S(r)}^P(x)$.

Let $v \in U$ and $t_\lambda := T_S(x-\lambda v)$, where $\lambda > 0$. Since $T_S(x) < \infty$, in view of the definition of T_S, for any $\varepsilon \in (0,r)$, there are $r_1 \in (r, r+\varepsilon)$, $s \in S$, and $u \in U$ such that $s = x+r_1u$. The convexity of U implies that

$$s \in S \cap (x-\lambda v+\lambda U+r_1U) \subset S \cap \big[x-\lambda v+(\lambda+r_1)U\big],$$

and hence $t_\lambda \leqslant \lambda+r_1 < \infty$. It follows from (3.8) that for $\lambda \in (0, \frac{\delta}{\|v\|+1})$,

$$\lambda+\varepsilon \geqslant \lambda+r_1-r \geqslant t_\lambda-r \geqslant \lambda\langle-\xi, v\rangle-\lambda^2\sigma\|v\|^2.$$

Since $\varepsilon > 0$ is arbitrary,

$$\lambda \geqslant \lambda\langle-\xi, v\rangle-\lambda^2\sigma\|v\|^2, \quad \forall\lambda \in \left(0, \frac{\delta}{\|v\|+1}\right).$$

Letting $\lambda \to 0+$ yields that $\langle-\xi, v\rangle \leqslant 1$. Therefore, $\Im_U(-\xi) \leqslant 1$.

For any $\eta > 0$, let

$$0 < \varepsilon < \min\left\{\frac{\eta^2}{(M^2\sigma+1)^2}, r^2\right\},$$

where $M = \sup_{v\in U}\|v\|$ and σ is the constant in (3.8). Then $\sqrt{\varepsilon} < r < r_1$. Since

$$s = x + r_1 u \in S \cap \bigl(x + \sqrt{\varepsilon}u + (r_1 - \sqrt{\varepsilon})U\bigr),$$

$T_S(x + \sqrt{\varepsilon}u) \leqslant r_1 - \sqrt{\varepsilon} < \infty$. It follows from (3.8) that for sufficiently small $\varepsilon > 0$,

$$\varepsilon - \sqrt{\varepsilon} \geqslant T_S(x + \sqrt{\varepsilon}u) - r \geqslant \sqrt{\varepsilon}\langle\xi, u\rangle - \varepsilon\sigma\|u\|^2 \geqslant \sqrt{\varepsilon}\langle\xi, u\rangle - \varepsilon\sigma M^2.$$

Therefore,

$$\inf_{w\in U}\langle\xi, w\rangle \leqslant \langle\xi, u\rangle \leqslant \bigl(1 + \sigma M^2\bigr)\sqrt{\varepsilon} - 1 < \eta - 1.$$

This verifies that $\inf_{w\in U}\langle\xi, w\rangle \leqslant -1$.

Therefore, $\Im_U(-\xi) = 1$.

(b) Let $\xi \in N^P_{S(r)}(x)$ be such that $\Im_U(-\xi) = 1$. Then there exist $\sigma, \delta > 0$ such that

$$\langle\xi, y - x\rangle \leqslant \sigma\|y - x\|^2, \tag{3.10}$$

for all $y \in S(r) \cap B(x;\delta)$. Since T_S is calm at x, we may assume that this δ is such that for some $k > 0$,

$$\bigl|T_S(y) - T_S(x)\bigr| \leqslant k\|y - x\|, \quad \forall y \in B(x;\delta). \tag{3.11}$$

Let $M := \sup_{u\in U}\|u\|$, $\sigma_1 := 2\sigma(1 + M^2k^2)$, and $\delta_1 := \frac{\delta}{2(1+Mk)}$. Now we prove that for all $y \in B(x;\delta_1)$,

$$T_S(y) - T_S(x) - \langle\xi, y - x\rangle \geqslant -\sigma_1\|y - x\|^2. \tag{3.12}$$

Now we assume that $t := T_S(y)$.

If $y \in B(x;\delta_1)$ and $T_S(y) > r$, by the definition of T_S, for any $\varepsilon \in (0, \frac{\delta}{2M+1})$, there are $t_1 \in (t, t + \varepsilon)$ and $q \in U$ such that $y + t_1 q \in S$. Take $z := y + (t_1 - r)q$. Then

$$z + rq = y + t_1 q \in S \cap (z + rU) \neq \emptyset.$$

This implies that $T_S(z) \leqslant r$. Moreover,

$$\|z - x\| \leqslant \|y - x\| + (t_1 - r)\|q\| \leqslant \|y - x\| + M(t_1 - r) \leqslant (1 + Mk)\|y - x\| + M\varepsilon < \delta.$$

This verifies that $z \in S(r) \cap B(x;\delta)$. It follows from (3.10) that

$$\begin{aligned} T_S(y) - T_S(x) - \langle\xi, y - x\rangle &= t - r - \langle\xi, y - z\rangle - \langle\xi, z - x\rangle \geqslant t - r - \langle\xi, y - z\rangle - \sigma\|z - x\|^2 \\ &= t - r + (t_1 - r)\langle\xi, q\rangle - \sigma\|z - x\|^2 \geqslant t - t_1 - \sigma\|z - x\|^2 \\ &\geqslant -\varepsilon - 2\sigma M^2(t_1 - r)^2 - 2\sigma\|y - x\|^2 \geqslant -\varepsilon - 2\sigma M^2(t + \varepsilon - r)^2 - 2\sigma\|y - x\|^2. \end{aligned}$$

Letting $\varepsilon \to 0+$, we obtain that

$$T_S(y) - T_S(x) - \langle\xi, y - x\rangle \geqslant -2\sigma M^2(t - r)^2 - 2\sigma\|y - x\|^2 \geqslant -2\sigma\bigl(1 + M^2k^2\bigr)\|y - x\|^2 = -\sigma_1\|y - x\|^2,$$

where the second inequality follows from (3.11).

If $y \in B(x;\delta_1)$ and $T_S(y) < r$, for any $\varepsilon \in (0, r - t)$, there is $t_\varepsilon \in (t, t + \varepsilon)$ such that

$$S \cap (y + t_\varepsilon U) \neq \emptyset.$$

Let $u \in U$ be such that

$$\langle-\xi, u\rangle > \Im_U(-\xi) - \varepsilon = 1 - \varepsilon$$

and let $z_1 := y - (r - t_\varepsilon)u$. Since $u \in U$ and since U is convex, we have

$$(r - t_\varepsilon)u + t_\varepsilon U \subset (r - t_\varepsilon)U + t_\varepsilon U \subset rU,$$

and hence

$$y + t_\varepsilon U = z_1 + (r - t_\varepsilon)u + t_\varepsilon U \subset z_1 + rU.$$

This implies that

$$\emptyset \neq (y + t_\varepsilon U) \cap S \subset (z_1 + rU) \cap S.$$

Therefore $T_S(z_1) \leqslant r$. Moreover, (3.11) implies that

$$\|z_1 - x\| \leqslant \|y - x\| + (r - t_\varepsilon)\|u\| \leqslant \|y - x\| + (r - t)M \leqslant (1 + kM)\|y - x\| \leqslant (1 + kM)\delta_1 < \delta.$$

This verifies that $z_1 \in S(r) \cap B(x;\delta)$. It follows from (3.10) that

$$\begin{aligned} T_S(y) - T_S(x) - \langle \xi, y - x\rangle &= t - r - \langle \xi, y - z_1\rangle - \langle \xi, z_1 - x\rangle \geqslant t - r - (r - t_\varepsilon)\langle \xi, u\rangle - \sigma\|z_1 - x\|^2 \\ &\geqslant t - r + (r - t_\varepsilon)(1 - \varepsilon) - \sigma\|z_1 - x\|^2 \\ &\geqslant -\varepsilon(1 + r - t) - 2\sigma\|u\|^2(t_\varepsilon - r)^2 - 2\sigma\|y - x\|^2 \\ &\geqslant -\varepsilon(1 + r - t) - 2\sigma M^2(t_\varepsilon - r)^2 - 2\sigma\|y - x\|^2. \end{aligned}$$

Letting $\varepsilon \to 0+$ and applying (3.11), we obtain the desired conclusion. □

Remark 3.1. The boundedness assumption of U cannot be removed. For example, let $U = X$ and $x_0 \notin S$. For every $x \in X$, since $S \cap (x + tU) = S$ for any $t > 0$, the definition of T_S yields $T_S(x) = 0$. Therefore $r = T_S(x_0) = 0$ and $S(r) = X$. It follows that

$$\{0\} = \partial^P T_S(x_0) \neq \emptyset = N^P_{S(r)}(x_0) \cap \big\{\xi \in X^*\colon\ \Im_U(-\xi) = 1\big\}.$$

Thus the conclusion of Theorem 3.2 does not hold.

4. Fréchet subdifferential of a minimal time function

Theorem 4.1. *Let $x \in S$. Then*

$$\partial^F T_S(x) = N^F_S(x) \cap \big\{\xi \in X^*\colon\ \Im_U(-\xi) \leqslant 1\big\}.$$

Proof. Let $\xi \in \partial^F T_S(x)$. Then for any $\sigma > 0$, there exists $\delta > 0$ such that

$$T_S(y) - T_S(x) - \langle \xi, y - x\rangle \geqslant -\sigma\|y - x\| \tag{4.1}$$

for all $y \in B(x;\delta)$. Since $T_S(\cdot) = 0$ on S and since $x \in S$, it follows that

$$\langle \xi, y - x\rangle \leqslant \sigma\|y - x\| \tag{4.2}$$

for all $y \in S \cap B(x;\delta)$. Therefore $\xi \in N^F_S(x)$.

Fix any $v \in U$. Let $t_\lambda := T_S(x - \lambda v)$, where $\lambda > 0$. Since $x \in S \cap (x - \lambda v + \lambda U) \neq \emptyset$, $t_\lambda \leqslant \lambda < \infty$. It follows from (4.1) that for sufficiently small $\lambda > 0$,

$$\lambda \geqslant t_\lambda \geqslant \lambda\langle -\xi, v\rangle - \lambda\sigma\|v\|,$$

which implies that $\langle -\xi, v\rangle \leqslant 1 + \sigma\|v\|$. Since $\sigma > 0$ and $v \in U$ are arbitrary, $\Im_U(-\xi) \leqslant 1$.

Conversely, let $\xi \in N^F_S(x)$ be such that $\Im_U(-\xi) \leqslant 1$. For any $\sigma > 0$, take $\sigma_0 \in (0, \frac{\sigma}{(1+M\|\xi\|)})$, where $M := \sup_{u\in U}\|u\|$. By the definition of Fréchet normal cone, there exists $\delta > 0$ such that

$$\langle \xi, y - x\rangle \leqslant \sigma_0\|y - x\|, \tag{4.3}$$

for all $y \in S \cap B(x;\delta)$. Let $\delta_0 := \frac{\delta}{2(1+M\|\xi\|)}$. Since $T_S(y) = 0$ for $y \in S$. It follows that

$$T_S(y) - \langle \xi, y - x\rangle \geqslant -\sigma\|y - x\|, \tag{4.4}$$

for all $y \in S \cap B(x;\delta_0)$.

Now we prove that (4.4) holds for all $y \in B(x;\delta_0) \setminus S$. Therefore, $\xi \in \partial^F T_S(x)$. If not, then there is $y_0 \notin S$ such that

$$\|y_0 - x\| < \delta_0 \quad \text{and} \quad T_S(y_0) < \langle \xi, y_0 - x\rangle - \sigma\|y_0 - x\|. \tag{4.5}$$

The latter implies that

$$T_S(y_0) \leqslant \|\xi\|\|y_0 - x\|. \tag{4.6}$$

Let $t := T_S(y_0)$. By the definition of T_S, for any $\varepsilon \in (0, \frac{\delta}{2M})$, there are $t_1 \in (0, t+\varepsilon)$, $s \in S$, and $u \in U$ such that $s = y_0 + t_1 u$. Thus (4.6) yields that

$$\begin{aligned}\|s-x\| &\leqslant \|y_0 - x\| + t_1\|u\| \leqslant \|y_0-x\| + (t+\varepsilon)M \leqslant \big(1+M\|\xi\|\big)\|y_0-x\| + \varepsilon M \\ &< \big(1+M\|\xi\|\big)\delta_0 + \varepsilon M < \delta.\end{aligned} \tag{4.7}$$

This verifies that $s \in S \cap B(x;\delta)$. Applying (4.3) and $\Im_U(-\xi) \leqslant 1$, we have

$$\begin{aligned}T_S(y_0) - \langle \xi, y_0 - x\rangle &= t - \langle \xi, y_0 - s\rangle - \langle \xi, s-x\rangle \geqslant t - \langle \xi, y_0 - s\rangle - \sigma_0\|s-x\| = t + t_1\langle \xi, u\rangle - \sigma_0\|s-x\| \\ &\geqslant -\varepsilon - \sigma_0\|s-x\| \geqslant -\varepsilon - \sigma_0\|u\|t_1 - \sigma_0\|y_0-x\| \geqslant -\varepsilon - \sigma_0 M(t+\varepsilon) - \sigma_0\|y_0 - x\| \\ &\geqslant -(1+\sigma_0 M)\varepsilon - \sigma_0\big(1+M\|\xi\|\big)\|y_0-x\| \geqslant -(1+\sigma_0 M)\varepsilon - \sigma\|y_0-x\|.\end{aligned}$$

Letting $\varepsilon \to 0+$ yields that

$$T_S(y_0) - \langle \xi, y_0 - x\rangle \geqslant -\sigma\|y_0-x\|,$$

which contradicts to (4.5). □

Theorem 4.2. *Let $x \notin S$ and $r := T_S(x) < \infty$. We have*

(a) $\partial^F T_S(x) \subset N^F_{S(r)}(x) \cap \{\xi \in X^*\colon \Im_U(-\xi) = 1\}$;
(b) *If T_S is calm at x, then*

$$N^F_{S(r)}(x) \cap \big\{\xi \in X^*\colon \Im_U(-\xi) = 1\big\} \subset \partial^F T_S(x).$$

Proof. (a) Let $\xi \in \partial^F T_S(x)$. For any $\sigma > 0$, there is $\delta > 0$ such that

$$T_S(y) - T_S(x) - \langle \xi, y-x\rangle \geqslant -\sigma\|y-x\|, \tag{4.8}$$

for all $y \in B(x;\delta)$. Since $T_S(\cdot) \leqslant r$ on $S(r)$ and $r = T_S(x)$, it follows that

$$\langle \xi, y-x\rangle \leqslant \sigma\|y-x\|, \tag{4.9}$$

for all $y \in S(r) \cap B(x;\delta)$, that is $\xi \in N^F_{S(r)}(x)$.

Let $v \in U$ and $t_\lambda := T_S(x - \lambda v)$, where $\lambda > 0$. Since $T_S(x) < \infty$, for any $\varepsilon > 0$, there are $r_1 \in (r, r+\varepsilon)$, $s \in S$, and $u \in U$ such that $s = x + r_1 u$. The convexity of U implies that

$$s \in S \cap (x - \lambda v + \lambda U + r_1 U) \subset S \cap \big(x - \lambda v + (\lambda + r_1)U\big),$$

and hence $t_\lambda \leqslant \lambda + r_1 < \infty$. It follows from (4.8) that

$$\lambda + \varepsilon \geqslant t_\lambda - r \geqslant \lambda\langle -\xi, v\rangle - \lambda\sigma\|v\|.$$

Since $\varepsilon > 0$ is arbitrary,

$$\lambda \geqslant \lambda\langle -\xi, v\rangle - \lambda\sigma\|v\|.$$

Since $\sigma > 0$ is arbitrary, $\langle -\xi, v\rangle \leqslant 1$. Therefore, $\Im_U(-\xi) \leqslant 1$.

For any $\eta > 0$, if $\sigma \in (0, \frac{\eta}{2M})$, $0 < \varepsilon < \min\{\frac{\eta^2}{4}, r^2, \frac{\delta^2}{M^2}\}$, where $M := \sup_{u\in U}\|u\|$. Then $0 < \sqrt{\varepsilon} < r < r_1$. Since

$$s = x + r_1 u \in S \cap \big(x + \sqrt{\varepsilon}u + (r_1 - \sqrt{\varepsilon})U\big),$$

and hence $T_S(x + \sqrt{\varepsilon}u) \leqslant r_1 - \sqrt{\varepsilon} < \infty$. Since $x + \sqrt{\varepsilon}u \in B(x;\delta)$, it follows from (4.8) that

$$\varepsilon - \sqrt{\varepsilon} \geqslant T_S(x + \sqrt{\varepsilon}u) - r \geqslant \sqrt{\varepsilon}\langle \xi, u\rangle - \sqrt{\varepsilon}\sigma\|u\| \geqslant \sqrt{\varepsilon}\langle \xi, u\rangle - \sqrt{\varepsilon}\sigma M;$$

that is,

$$\langle -\xi, u\rangle \geqslant 1 - \sqrt{\varepsilon} - \sigma M = 1 - \sqrt{\varepsilon} - \sigma M > 1 - \eta.$$

Since $\eta > 0$ is arbitrary, $\Im_U(-\xi) = 1$.

(b) Let $\xi \in N^F_{S(r)}(x)$ be such that $\Im_U(-\xi) = 1$. Since T_S is calm at x, there are $\delta > 0$ and $k > 0$ such that

$$\big|T_S(y) - T_S(x)\big| \leqslant k\|y-x\|, \quad \forall y \in B(x;\delta). \tag{4.10}$$

For any $\sigma > 0$, take $\sigma_0 \in (0, \frac{\sigma}{1+kM})$. Since $\xi \in N^F_{S(r)}(x)$, we may assume that $\delta > 0$ is such that

$$\langle \xi, y - x\rangle \leqslant \sigma_0 \|y - x\|, \quad \forall y \in S(r) \cap B(x; \delta). \tag{4.11}$$

Let $\sigma_1 := 2\sigma(1 + M^2k^2)$ and $\delta_1 := \frac{\delta}{2(1+Mk)}$. Now we prove that for all $y \in B(x; \delta_1)$,

$$T_S(y) - T_S(x) - \langle \xi, y - x\rangle \geqslant -\sigma_1 \|y - x\|. \tag{4.12}$$

Now we assume that $t := T_S(y)$.

If $T_S(y) > r$, by the definition of T_S, for any $\varepsilon \in (0, \frac{M}{2\delta})$, there are $t_1 \in (t, t + \varepsilon)$, $p \in S$, and $q \in U$ such that $p = y + t_1 q$. Take $z := y + (t_1 - r)q$. Then

$$z + rq = y + t_1 q \in S \cap (z + rU) \neq \emptyset.$$

This implies that $T_S(z) \leqslant r$. Moreover, (4.10) implies that

$$\|z - x\| \leqslant \|y - x\| + (t_1 - r)\|q\| \leqslant \|y - x\| + M(t_1 - r) \leqslant (1 + Mk)\|y - x\| + M\varepsilon < \delta.$$

This verifies that $z \in S(r) \cap B(x; \delta)$. By virtue of (4.11),

$$\begin{aligned} T_S(y) - T_S(x) - \langle \xi, y - x\rangle &= t - r - \langle \xi, y - z\rangle - \langle \xi, z - x\rangle \geqslant t - r - \langle \xi, y - z\rangle - \sigma_0\|z - x\| \\ &= t - r + (t_1 - r)\langle \xi, q\rangle - \sigma_0\|z - x\| \geqslant -\varepsilon - \sigma_0\|z - x\| \\ &\geqslant -\varepsilon - \sigma_0 M(t_1 - r) - \sigma_0\|y - x\| \geqslant -(1 + \sigma_0 M)\varepsilon - \sigma_0(1 + kM)\|y - x\| \\ &\geqslant -(1 + \sigma_0 M)\varepsilon - \sigma\|y - x\|. \end{aligned}$$

Letting $\varepsilon \to 0+$, we have

$$T_S(y) - T_S(x) - \langle \xi, y - x\rangle \geqslant -\sigma\|y - x\|.$$

If $T_S(y) < r$, for any $\varepsilon \in (0, r - t)$, the definition of T_S implies the existence of $t_\varepsilon \in (t, t + \varepsilon)$ such that $S \cap (y + t_\varepsilon U) \neq \emptyset$. Since $\Im_U(-\xi) = 1$, there is $u \in U$ such that

$$\langle -\xi, u\rangle > 1 - \varepsilon.$$

Let $z := y - (r - t_\varepsilon)u$. Since U is convex,

$$(r - t_\varepsilon)u + t_\varepsilon U \subset (r - t_\varepsilon)U + t_\varepsilon U \subset rU,$$

and hence

$$\emptyset \neq (y + t_\varepsilon U) \cap S \subset (z + rU) \cap S.$$

This implies that $T_S(z) \leqslant r$. Moreover, (4.10) implies that

$$\|z - x\| \leqslant \|y - x\| + (r - t_\varepsilon)\|u\| \leqslant \|y - x\| + (r - t)M \leqslant (1 + kM)\|y - x\| \leqslant (1 + kM)\delta_1 < \delta.$$

Thus $z \in S(r) \cap B(x; \delta)$. It follows from (4.11) that

$$\begin{aligned} T_S(y) - T_S(x) - \langle \xi, y - x\rangle &= t - r - \langle \xi, y - z\rangle - \langle \xi, z - x\rangle \geqslant t - r - (r - t_\varepsilon)\langle \xi, u\rangle - \sigma_0\|z - x\| \\ &\geqslant -\varepsilon(1 + r - t) - \sigma_0\|z - x\| \geqslant -\varepsilon(1 + r - t) - \sigma_0\|u\|(t_\varepsilon - r) - \sigma_0\|y - x\| \\ &\geqslant -\varepsilon(1 + r - t) - \sigma_0 M(t_\varepsilon - r) - \sigma_0\|y - x\|. \end{aligned}$$

Letting $\varepsilon \to 0+$, we have

$$T_S(y) - T_S(x) - \langle \xi, y - x\rangle \geqslant -\sigma_0 M(t - r) - \sigma_0\|y - x\| \geqslant -\sigma_0(1 + kM)\|y - x\| = -\sigma\|y - x\|.$$

This completes the proof. $\square$

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